J. Math. Anal. Appl. 392 (2012) 103–110

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Characterizations of perturbations of spectra of 2×2 upper triangular operator matrices ☆

Shifang Zhang [a], Zhaoqi Wu [b,*]

[a] School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, PR China
[b] Department of Mathematics, Nanchang University, Nanchang 330031, PR China

ARTICLE INFO

Article history:
Received 14 September 2011
Available online 11 April 2012
Submitted by Lawrence A. Fialkow

Keywords:
Hilbert space
Operator matrices
Spectra

ABSTRACT

When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C the operator acting on the infinite dimensional separable Hilbert space $H \oplus K$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, we first give some necessary and sufficient conditions for M_C to be a left invertible operator (an upper semi-Weyl, upper semi-Fredholm) operator for some $C \in B(K, H)$, which extend the corresponding results in Cao et al. (2006) [4], Cao and Meng (2005) [5], Hwang and Lee (2001) [12] and Li and Du (2006) [15]. Then we present some counter-examples.

1. Introduction

It is well known that if H is a Hilbert space, T is a bounded linear operator defined on H and H_1 is an invariant closed subspace of T, then T can be represented in the form of

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : H_1 \oplus H_1^{\perp} \to H_1 \oplus H_1^{\perp},$$

which motivated the interest in 2×2 upper-triangular operator matrices. For recent investigations on this subject, see Refs. [1–24].

Throughout this paper, let H and K be the complex infinite dimensional separable Hilbert spaces and let $B(H, K)$ be the set of all bounded linear operators from H into K. For simplicity, we also write $B(H, H)$ as $B(H)$.

For $T \in B(H, K)$, we use $R(T)$ and $N(T)$ to denote the range and kernel of T, respectively. Denote $\alpha(T) = \dim N(T)$, $\beta(T) = \dim K/R(T)$ and $d(T) = \dim K/\overline{R(T)}$. It is well-known that $\beta(T) < \infty$ implies that $R(T)$ is closed, and $\beta(T) = d(T)$ when $R(T)$ is closed. It is worth mentioning that in [4,5], the authors used $n(A)$ and $d(A)$ instead of $\alpha(A)$ and $\beta(A)$ respectively.

For $T \in B(H, K)$, if $R(T)$ is closed and $\alpha(T) < \infty$, then T is called an upper semi-Fredholm operator and if $\beta(T) < \infty$, then T is called a lower semi-Fredholm operator. If $T \in B(H, K)$ is either upper or lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

We denote the set of all invertible (resp., left invertible, right invertible) operators from H into K by $G(H, K)$ (resp., $G_l(H, K)$, $G_r(H, K)$), the set of all Fredholm (resp., upper semi-Fredholm, lower semi-Fredholm) operators from H into K by $\Phi(H, K)$ (resp., $\Phi_+(H, K)$, $\Phi_-(H, K)$), the set of all Weyl (resp., upper semi-Weyl, lower semi-Weyl) operators from H into K by $\Phi_0(H, K)$ (resp., $\Phi_+^-(H, K)$, $\Phi_-^+(H, K)$). The spectrum, left spectrum, right spectrum, essential spectrum, upper semi-Fredholm spectrum, lower semi-Fredholm spectrum, Weyl spectrum, upper semi-Weyl spectrum, lower semi-Fredholm

☆ This work is supported by the NSF of China (Grant Nos. 11171301, 11171066).
* Corresponding author.
E-mail address: wuzhaoqi_conquer@163.com (Z. Wu).

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.03.031

spectrum of an operator $T \in B(H)$ are respectively denoted by $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_e(T)$, $\sigma_{SF+}(T)$, $\sigma_{SF-}(T)$, $\sigma_w(T)$, $\sigma_{aw}(T)$ and $\sigma_{sw}(T)$. The definitions of the above-mentioned concepts can be found in [4,5,7,14–16,22].

Henceforth, for $A \in B(H)$, $B \in B(K)$ and $C \in B(K,H)$, we put $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. It is clear that $M_C \in B(H \oplus K)$.

In [10], the authors showed that M_C is invertible for some $C \in B(K,H)$ if and only if A is left invertible, B is right invertible and $\beta(A) = \alpha(B)$. Han et al. extended the above result to the general Banach space case in [11]. In [4], Cao et al. gave a necessary and sufficient condition for M_C to be an upper semi-Fredholm operator (resp., a lower semi-Fredholm operator, a Fredholm operator) for some $C \in B(K,H)$ and characterized the intersection of the upper semi- Fredholm spectrum and the lower semi-Fredholm spectrum of M_C. In [5], Cao et al. obtained a necessary and sufficient condition for M_C to be an upper semi-Weyl operator (resp., a lower semi-Weyl operator, a Weyl operator) for some $C \in B(K,H)$ and characterized the intersection of the upper semi-Weyl spectrum, the lower semi-Weyl spectrum and the Weyl spectrum of M_C.

In Section 2 of this paper, we extend all the results mentioned above by the same technique and similar proofs, and the statements of the results seem much more simpler. Moreover, some counter-examples are presented in Section 3.

2. The characterizations of M_C to be a left invertible (an upper semi-Weyl, an upper semi-Fredholm) operator for some $C \in B(K,H)$

Hwang and Lee [12] provided a necessary and sufficient condition for M_C to be left invertible operator for some $C \in B(K,H)$ and characterized the intersection of the left spectrum, the right spectrum and the spectrum of M_C. One of their main results is as follows:

For a given pair (A,B) of operators, M_C is bounded below for some $C \in B(K,H)$ if and only if A is bounded below and

$$\begin{cases} \alpha(B) \le d(A) & \text{if } R(B) \text{ is closed}, \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed}. \end{cases}$$

This result is interesting and correct. However, the proof does not work well because the following claim was used, which is not always true:

If

$$\alpha(B) + d(M_C) = d(A) + d\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\right),$$

then

$$\alpha(B) \le d(A)$$

holds since

$$d(M_C) \ge d\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\right).$$

In fact, it is easy to see that when

$$d(M_C) = d\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\right) = \infty,$$

the above claim fails. If we add the additional condition $d(M_C) < \infty$, then the claim can be true.

By using another approach, which seems much simpler and without making use of the above claim, we prove the above theorem. Furthermore, we present other characterizations for M_C to be a left invertible operator.

The following result is an extension of Theorem 1 in [12].

Theorem 2.1. *For a given pair (A,B) of operators, the following statements are equivalent:*

(i) *M_C is left invertible for some $C \in B(K,H)$,*

(ii) *A is left invertible and* $\begin{cases} \beta(A) = \infty \\ \text{or } B \in \Phi_+(K) \text{ and } \alpha(B) \le \beta(A), \end{cases}$

(iii) *A is left invertible and* $\begin{cases} \alpha(B) \le d(A) & \text{if } R(B) \text{ is closed}, \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed}, \end{cases}$

(iv) *there exists some left invertible operator $C \in B(K,H)$ such that M_C is left invertible.*

Proof. (iv) ⇒ (i) is obvious.

(i) ⇒ (ii) If M_C is left invertible, then it is easy to see that A is left invertible. In this case, if $\beta(A) \ne \infty$, then $A \in \Phi(H)$, and so $B \in \Phi_+(K)$. From the above argument, in order to show (i) ⇒ (ii), we only need to prove that if M_C is left invertible, then $\alpha(B) \le \beta(A)$. For this, it is sufficient to prove that if A is left invertible with $\alpha(B) > \beta(A)$, then M_C is not left invertible. Since $\alpha(B) > \beta(A)$, we have $\beta(A) < \infty$. We now consider the following two cases.

Case (I) Suppose $N(C) \cap N(B) \ne \{0\}$. Then there exists some $y \in K$ such that $0 \ne y \in N(C) \cap N(B)$. Obviously, $M_C \begin{pmatrix} 0 \\ y \end{pmatrix} = 0$. This implies that M_C is not left invertible.

Case (II) Suppose $N(C) \cap N(B) = \{0\}$. Then

$$\dim C(N(B)) = \alpha(B) > \beta(A).$$

Thus,

$$RC(N(B)) \cap R(A) \neq \{0\}.$$

Let

$$0 \neq z \in RC(N(B)) \cap R(A).$$

Then there exist some $x \in X$ and $y \in K$ such that

$$Ax = Cy = z \quad \text{and} \quad By = 0.$$

Direct calculation shows that $M_C \begin{pmatrix} -x \\ y \end{pmatrix} = 0$. This implies that M_C is not left invertible.

(ii) $\Rightarrow$ (iv) If A is left invertible and $\beta(A) = \infty$, then we define an operator

$$Q : K \to H \quad \text{by } Q = \begin{pmatrix} T \\ 0 \end{pmatrix} : K \to \begin{pmatrix} R(A)^{\perp} \\ R(A) \end{pmatrix},$$

where T is an invertible operator form K onto $R(A)^{\perp}$. It is obvious that Q is left invertible. Moreover, we can claim that M_Q is left invertible.

On the other hand, if A is left invertible, $B \in \Phi_-(K)$ and $\alpha(B) \le \beta(A)$, then we define an operator

$$Q : K \to H \quad \text{by } Q = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \begin{pmatrix} N(B) \\ N(B)^{\perp} \end{pmatrix} \to \begin{pmatrix} R(A)^{\perp} \\ R(A) \end{pmatrix},$$

where C_1 is an isomorphism and C_2 is an unitary operator. It is obvious that Q is left invertible. Moreover, we can claim that M_Q is left invertible.

It remains to prove (ii) $\Leftrightarrow$ (iii). Note that if $R(A)$ is closed, then

$$\begin{cases} \beta(A) = \infty \\ \text{or } B \in \Phi_+(K) \text{ and } \alpha(B) \le \beta(A), \end{cases}$$

$$\Leftrightarrow \begin{cases} R(B) \text{ is closed and } \beta(A) = \infty, \\ \text{or } R(B) \text{ is not closed and } \beta(A) = \infty, \\ \text{or } \alpha(B) < \infty, \alpha(B) \le \beta(A) \text{ and } R(B), \text{ is closed.} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha(B) \le \beta(A) & \text{if } R(B) \text{ is closed,} \\ \beta(A) = \infty, & \text{if } R(B) \text{ is not closed,} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha(B) \le d(A) & \text{if } R(B) \text{ is closed,} \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

So (ii) $\Leftrightarrow$ (iii) This completes the proof. □

By duality, we have

Theorem 2.2. *For a given pair (A, B) of operators, the following statements are equivalent:*

(i) *M_C is right invertible for some $C \in B(K, H)$,*
(ii) *B is right invertible and* $\begin{cases} \alpha(B) = \infty \\ \text{or } A \in \Phi_-(H) \text{ and } \alpha(B) \ge \beta(A), \end{cases}$
(iii) *B is right invertible and*

$$\begin{cases} \beta(A) \le \alpha(B) & \text{if } R(A) \text{ is closed}, \\ \alpha(B) = \infty, & \text{if } R(A) \text{ is not closed}. \end{cases}$$

(iv) *there exists some right invertible operator $C \in B(K, H)$ such that M_C is right invertible.*

As a direct application of Theorems 2.1 and 2.2, the following two corollaries can be derived which give the characterizations of $\bigcap_{C \in B(K,H)} \sigma_a(M_C)$ and $\bigcap_{C \in B(K,H)} \sigma_d(M_C)$ respectively. Note that they are different from the corresponding forms in [12].

Corollary 2.3. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C \in G_l(K,H)} \sigma_l(M_C) = \bigcap_{C \in B(K,H)} \sigma_l(M_C) = \sigma_l(A) \cup \{\lambda \in \sigma_{SF+}(B) : \beta(A - \lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \alpha(B - \lambda) > \beta(A - \lambda)\}.$$

Corollary 2.4. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C\in G_r(K,H)} \sigma_r(M_C) = \bigcap_{C\in B(K,H)} \sigma_r(M_C) = \sigma_r(B) \cup \{\lambda \in \sigma_{SF-}(A) : \alpha(B-\lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \beta(A-\lambda) > \alpha(B-\lambda)\}.$$

In [5], Cao et al. have shown that, for a given pair (A, B) of operators, $M_C \in \Phi_+^-(H \oplus K)$ for some $C \in B(K, H)$ if and only if $A \in \Phi_+(H)$ and

$$\begin{cases} \alpha(B) < \infty \text{ and } \alpha(A) + \alpha(B) \le \beta(A) + \beta(B) \text{ or } \alpha(B) = \beta(A) = \infty, & \text{if } R(B) \text{ is closed,} \\ \beta(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Now we generalize this result to the following theorem, using an alternative approach based on matrix representation of operators with respect to the space decomposition, which enables us to simplify the proof.

Theorem 2.5. *For a given pair (A, B) of operators, the following statements are equivalent:*

(i) *there exists some $C \in B(K, H)$ such that $M_C \in \Phi_+^-(H \oplus K)$,*

(ii) *$A \in \Phi_+(H)$ and $\begin{cases} \beta(A) = \infty, \\ \text{or } B \in \Phi_+(K) \text{ and } \alpha(A) + \alpha(B) \le \beta(A) + \beta(B), \end{cases}$*

(iii) *$A \in \Phi_+(H)$ and*

$$\begin{cases} \alpha(B) < \infty \text{ and } \alpha(A) + \alpha(B) \le \beta(A) + \beta(B) \text{ or } \alpha(B) = \beta(A) = \infty, & \text{if } R(B) \text{ is closed}, \\ \beta(A) = \infty, & \text{if } R(B) \text{ is not closed}, \end{cases}$$

(iv) *there exists some $Q \in G_l(K, H)$ such that $M_Q \in \Phi_+^-(H \oplus K)$.*

Proof. (iv) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii) Suppose that $M_C \in \Phi_+^-(H \oplus K)$ for some $C \in B(K, H)$. Then it is easy to show that $A \in \Phi_+(H)$. If $\beta(A) = \infty$, the desired result is obtained. On the other hand, if $\beta(A) < \infty$, then we have that $A \in \Phi(H)$. Also since $M_C \in \Phi_+(H \oplus K)$, it is not hard to show that $B \in \Phi_+(K)$. In this case, observe that

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Then $\operatorname{ind}(M_C) = \operatorname{ind}(A) + \operatorname{ind}(B) \le 0$. Accordingly, $B \in \Phi_+(K)$ and $\alpha(A) + \alpha(B) \le \beta(A) + \beta(B)$.

(ii) $\Rightarrow$ (iv) To show this, we consider the following two cases.

Case (I) Suppose that $A \in \Phi_+(H)$, $B \in \Phi_+(K)$ and $\alpha(A)+\alpha(B) \le \beta(A)+\beta(B)$. Then it is easy to show that $M_C \in \Phi_+(H\oplus K)$ with $\operatorname{ind}(M_C) = \operatorname{ind}(A) + \operatorname{ind}(B) \le 0$ for every $C \in B(K, H)$, which means that $M_Q \in \Phi_+^-(H \oplus K)$ for every $Q \in G_l(K, H)$.

Case (II) Suppose that that $A \in \Phi_+(H)$ and $\beta(A) = \infty$. Since $R(A)$ is closed and $\beta(A) = \infty$, there exist subspaces M and N of $R(A)^\perp$ such that $R(A)^\perp = M \oplus N$ and $\dim(M) = \dim(N) = \infty$. Let $T \in G(K, M)$ and define an operator

$$Q : K \to H \quad \text{by } Q = \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix} : K \to \begin{pmatrix} M \\ N \\ R(A) \end{pmatrix}.$$

Obviously, $Q \in G_l(K, H)$. Now, we claim that

$$M_Q \in \Phi_+^-(H \oplus K) \quad \text{and} \quad \alpha(A) = \alpha(M_Q).$$

In fact, if $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_Q)$, then $Ax + Qy = 0$ and $By = 0$. Thus $Ax = -Qy \in R(A) \cap M \subseteq R(A) \cap R(A)^\perp$, and hence $Ax = 0$ and $Qy = 0$. But since that $Qy = Ty = 0$, it follows that $y = 0$. Now it is clear that $N(M_Q) \subseteq N(A) \oplus \{0\}$, and so $N(M_Q) = N(A) \oplus \{0\}$. Thus, we have that $\alpha(M_Q) = \alpha(A) < \infty$. Next, we will prove that $R(M_Q)$ is closed. For this, suppose that $M_Q \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$, then $Ax_n + Qy_n \to x$ and $By_n \to y$. Thus both $\{Ax_n\}$ and $\{Qy_n\}$ are Cauchy sequences. And hence $\{y_n\}$ is a Cauchy sequence from the definition of operator Q. Since $R(A)$ and K are closed, it follows that there exist $x_0 \in H$ and $y_0 \in K$ such that $Ax_n \to Ax_0$ and $y_n \to y_0$. Therefore

$$M_Q \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \in R(M_Q),$$

which implies that $R(M_Q)$ is closed. Hence $M_Q \in \Phi_+(H\oplus K)$. Furthermore, since $\dim(N) = \infty$, it is obvious that $\beta(M_Q) = \infty$, and so $M_Q \in \Phi_+^-(H \oplus K)$.

(ii) $\Leftrightarrow$ (iii) This follows immediately by noting the following equivalent relations

$$\begin{cases}\beta(A)=\infty\\ \text{or } B\in\Phi_+(K) \text{ and } \alpha(A)+\alpha(B)\le\beta(A)+\beta(B).\end{cases}$$

$$\Leftrightarrow\begin{cases}R(B) \text{ is closed and } \beta(A)=\infty,\\ \text{or } R(B) \text{ is not closed and } \beta(A)=\infty,\\ \text{or } \alpha(B)<\infty, R(B) \text{ is closed and } \alpha(A)+\alpha(B)\le\beta(A)+\beta(B).\end{cases}$$

$$\Leftrightarrow\begin{cases}\alpha(B)<\infty, R(B) \text{ is closed and } \beta(A)=\infty,\\ \text{or } R(B) \text{ is closed and } \alpha(B)=\beta(A)=\infty,\\ \text{or } R(B) \text{ is not closed and } \beta(A)=\infty,\\ \text{or } \alpha(B)<\infty, R(B) \text{ is closed and } \alpha(A)+\alpha(B)\le\beta(A)+\beta(B).\end{cases}$$

$$\Leftrightarrow\begin{cases}R(B) \text{ is closed and } \alpha(B)=\beta(A)=\infty,\\ \text{or } R(B) \text{ is not closed and } \beta(A)=\infty,\\ \text{or } \alpha(B)<\infty, R(B) \text{ is closed and } \alpha(A)+\alpha(B)\le\beta(A)+\beta(B).\end{cases}$$

$$\Leftrightarrow\begin{cases}\alpha(B)<\infty \text{ and } \alpha(A)+\alpha(B)\le\beta(A)+\beta(B) \text{ or } \alpha(B)=\beta(A)=\infty, & \text{if } R(B) \text{ is closed},\\ \beta(A)=\infty, & \text{if } R(B) \text{ is not closed}.\end{cases}$$

Thus the theorem is proved. □

By duality, we have:

Theorem 2.6. *For a given pair (A, B) of operators, the following statements are equivalent:*

(i) *there exists some $C\in B(K,H)$ such that $M_C\in\Phi_-^+(H\oplus K)$,*

(ii) *$B\in\Phi_-(K)$ and* $\begin{cases}\alpha(B)=\infty,\\ \text{or } A\in\Phi_-(H) \text{ and } \alpha(A)+\alpha(B)\ge\beta(A)+\beta(B),\end{cases}$

(iii) *$B\in\Phi_-(K)$ and*

$$\begin{cases}\beta(A)<\infty \text{ and } \alpha(A)+\alpha(B)\ge\beta(A)+\beta(B) \text{ or } \beta(A)=\alpha(B)=\infty, & \text{if } R(A) \text{ is closed},\\ \alpha(B)=\infty, & \text{if } R(A) \text{ is not closed},\end{cases}$$

(iv) *there exists some $Q\in G_r(K,H)$ such that $M_Q\in\Phi_-^+(H\oplus K)$.*

The most important work in [5,15] is to characterize the sets $\bigcap_{C\in B(K,H)}\sigma_{aw}(M_C)$, $\bigcap_{C\in B(K,H)}\sigma_{sw}(M_C)$, $\bigcap_{C\in G_l(K,H)}\sigma_{aw}(M_C)$ and $\bigcap_{C\in G_r(K,H)}\sigma_{sw}(M_C)$. That is,

$$\bigcap_{C\in G_l(K,H)}\sigma_{aw}(M_C)=\bigcap_{C\in B(K,H)}\sigma_{aw}(M_C)=\sigma_{SF+}(A)\cup\Phi_{lw}(A,B)\cup\Upsilon_{lw}(A,B)\cup\Psi_l(A,B)$$

where

$$\Phi_{lw}(A,B)=\{\lambda\in\mathbb{C}: R(B-\lambda) \text{ is closed and } \alpha(B-\lambda)+\alpha(A-\lambda)>\beta(B-\lambda)+\beta(A-\lambda)\},$$
$$\Upsilon_{lw}(A,B)=\{\lambda\in\mathbb{C}: R(B-\lambda) \text{ is closed and } \alpha(B-\lambda)=\beta(B-\lambda)=\infty, \beta(A-\lambda)<\infty\},$$
$$\Psi_l(A,B)=\{\lambda\in\mathbb{C}: R(B-\lambda) \text{ is not closed and } \beta(A-\lambda)<\infty\}$$

and

$$\bigcap_{C\in G_r(K,H)}\sigma_{sw}(M_C)=\bigcap_{C\in B(K,H)}\sigma_{sw}(M_C)=\sigma_{SF-}(A)\cup\Phi_{rw}(A,B)\cup\Upsilon_{rw}(A,B)\cup\Psi_r(A,B)$$

where

$$\Phi_{rw}(A,B)=\{\lambda\in\mathbb{C}: R(A-\lambda) \text{ is closed}, \alpha(B-\lambda)+\alpha(A-\lambda)<\beta(B-\lambda)+\beta(A-\lambda)\},$$
$$\Upsilon_{rw}(A,B)=\{\lambda\in\mathbb{C}: R(A-\lambda) \text{ is closed and } \alpha(A-\lambda)=\beta(A-\lambda)=\infty, \alpha(B-\lambda)<\infty\},$$
$$\Psi_r(A,B)=\{\lambda\in\mathbb{C}: R(A-\lambda) \text{ is not closed and } \alpha(B-\lambda)<\infty\}.$$

Remark. Note that for operator $T\in B(K,H)$, if $R(T)$ is closed, then $\beta(T)=d(T)$. Also, if $\beta(T)\ne d(T)$, then $d(T)<\beta(T)=\infty$ and $R(T)$ is not closed, and thus $0\in\sigma_{SF-}(T)\cap\sigma_{SF+}(T)$. Therefore $\beta(A-\lambda)$ and $\beta(B-\lambda)$ in the above equalities can be replaced by $d(A-\lambda)$ and $d(B-\lambda)$, respectively.

As a direct application of Theorems 2.5 and 2.6, we can also characterize the sets $\bigcap_{C\in B(K,H)}\sigma_{aw}(M_C)$ and $\bigcap_{C\in B(K,H)}\sigma_{sw}(M_C)$ in other simple forms, respectively.

Corollary 2.7. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C\in G_l(K,H)} \sigma_{aw}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{aw}(M_C) = \sigma_{SF+}(A) \cup \{\lambda \in \sigma_{SF+}(B) : \beta(A-\lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) > \beta(A-\lambda) + \beta(B-\lambda)\}.$$

Corollary 2.8. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C\in G_r(K,H)} \sigma_{sw}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{sw}(M_C) = \sigma_{SF-}(B) \cup \{\lambda \in \sigma_{SF-}(A) : \alpha(B-\lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) < \beta(A-\lambda) + \beta(B-\lambda)\}.$$

Combining Corollaries 2.7 and 2.8, we rewrite Corollary 3.7 in [5] in the Hilbert space case.

Corollary 2.9. *For a given pair (A, B) of operators, we have*

$$\bigcap_{C\in B(K,H)} \sigma_w(M_C) = \sigma_{SF+}(A) \cup \sigma_{SF-}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)\}.$$

Proof. Note that $\{\lambda \in \sigma_{SF-}(A) : \alpha(B-\lambda) < \infty\} \setminus (\sigma_{SF+}(A) \cup \sigma_{SF-}(B)) \subseteq \{\lambda \in \mathbb{C} : \alpha(A-\lambda)+\alpha(B-\lambda) < \beta(A-\lambda)+\beta(B-\lambda)\}$ and that $\{\lambda \in \sigma_{SF+}(B) : \beta(A-\lambda) < \infty\} \setminus (\sigma_{SF+}(A) \cup \sigma_{SF-}(B)) \subseteq \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) > \beta(A-\lambda) + \beta(B-\lambda)\}$, the desired result follows from Corollaries 2.3 and 2.4. □

Similar to Theorem 2.5, we have the following result, which extends Theorem 2.2 in [4].

Theorem 2.10. *For a given pair (A, B) of operators, the following statements are equivalent:*

(i) *there exists some $C \in B(K, H)$ such that $M_C \in \Phi_+(H \oplus K)$,*

(ii) *$A \in \Phi_+(H)$ and $\begin{cases} \beta(A) = \infty \\ \text{or } B \in \Phi_+(K), \end{cases}$*

(iii) *$A \in \Phi_+(H)$ and*

$$\begin{cases} \alpha(B) < \infty \text{ or } \alpha(B) = \beta(A) = \infty, & \text{if } R(B) \text{ is closed}, \\ \beta(A) = \infty, & \text{if } R(B) \text{ is not closed}, \end{cases}$$

(iv) *there exists some $Q \in G_l(K, H)$ such that $M_Q \in \Phi_+(H \oplus K)$.*

Proof. Since the proof of (i) ⇔ (ii) ⇔ (iv) is quite similar to the proof of (i) ⇔ (ii) ⇔ (iv) in Theorem 2.5, here we only show that (ii) ⇔ (iii). In fact, this can be seen from the following equivalent relations

$$\begin{cases} \beta(A) = \infty \\ \text{or } B \in \Phi_+(K) \end{cases}$$

$$\Leftrightarrow \begin{cases} R(B) \text{ is closed and } \beta(A) = \infty, \\ \text{or } R(B) \text{ is not closed and } \beta(A) = \infty, \\ \text{or } \alpha(B) < \infty \text{ and } R(B) \text{ is closed.} \end{cases}$$

$$\Leftrightarrow \begin{cases} R(B) \text{ is closed and } \alpha(B) = \beta(A) = \infty, \\ \text{or } R(B) \text{ is not closed and } \beta(A) = \infty, \\ \text{or } \alpha(B) < \infty \text{ and } R(B) \text{is closed.} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha(B) < \infty \text{ or } \alpha(B) = \beta(A) = \infty, & \text{if } R(B) \text{ is closed}, \\ \beta(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Thus the theorem is proved. □

By duality, we have the following result.

Theorem 2.11. *For a given pair (A, B) of operators, the following statements are equivalent:*

(i) *there exists some $C \in B(K, H)$ such that $M_C \in \Phi_-(H \oplus K)$,*

(ii) *$B \in \Phi_-(K)$ and $\begin{cases} \alpha(B) = \infty \\ \text{or } A \in \Phi_-(H), \end{cases}$*

(iii) *$B \in \Phi_-(K)$ and*

$$\begin{cases} \beta(A) < \infty \text{ or } \beta(A) = \alpha(B) = \infty, & \text{if } R(A) \text{ is closed}, \\ \alpha(B) = \infty, & \text{if } R(A) \text{ is not closed}, \end{cases}$$

(iv) *there exists some $Q \in G_r(K, H)$ such that $M_Q \in \Phi_+(H \oplus K)$.*

From Theorems 2.10 and 2.11, the next two results follows immediately.

Corollary 2.12. *For a given pair* (A, B) *of operators, we have*

$$\bigcap_{C\in G_l(K,H)} \sigma_{SF+}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{SF+}(M_C) = \sigma_{SF+}(A) \cup \{\lambda \in \sigma_{SF+}(B) : \beta(A-\lambda) < \infty\}.$$

Corollary 2.13. *For a given pair* (A, B) *of operators, we have*

$$\bigcap_{C\in G_r(K,H)} \sigma_{SF-}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{SF-}(M_C) = \sigma_{SF-}(B) \cup \{\lambda \in \sigma_{SF-}(A) : \alpha(B-\lambda) < \infty\}.$$

3. Some counter-examples

In this section, we present some counter-examples to show that some results in [1,9] are not always true.

Let X, Y be Banach spaces, and $A \in B(X)$, $B \in B(Y)$ and $C \in B(Y, X)$ be arbitrary operators. In [9, Lemma 3.2] says that: if M_C is an upper semi-Fredholm operator with $\operatorname{ind}(M_C) \le 0$, then A is an upper semi-Fredholm operator and

$$\begin{cases} \alpha(B) < \infty \text{ and } \operatorname{ind}(A) + \operatorname{ind}(B) \le 0, \text{ or} \\ \alpha(B) = \beta(A) = \infty. \end{cases} \tag{1}$$

The following example illustrates that the above claim is not always true.

Example 3.1. Let $H = K = \ell^2$. Define the operators A, B and C by

$$\begin{aligned} &A : H \to H, \qquad \{x_1, x_2, x_3, \ldots\} \mapsto \{0, x_1, 0, x_2, \ldots\}, \\ &B : K \to K, \qquad \{x_1, x_2, x_3, \ldots\} \mapsto \left\{0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, \ldots\right\}, \\ &C : K \to H, \qquad \{x_1, x_2, x_3, \ldots\} \mapsto \{x_1, 0, x_2, 0, x_3, \ldots\}. \end{aligned}$$

Consider the operator

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2.$$

Then we have

(i) from Theorem 2.1, we know that $M_C \in \Phi_+^-(\ell^2 \oplus \ell^2)$;
(ii) B is injective and $\operatorname{ind}(B)$ makes no sense since B is not a semi-Fredholm operator.

Therefore, from (i) and (ii), we know that Lemma 3.2 in [9] does not always hold. However, it is easy to show that if M_C is an upper semi-Fredholm operator with $\operatorname{ind}(M_C) \le 0$, then A is upper semi-Fredholm and

$$\begin{cases} B \in \Phi_+(Y) \text{ and } \operatorname{ind}(A) + \operatorname{ind}(B) \le 0, \text{ or} \\ \beta(A) = \infty. \end{cases}$$

It is claimed in [9] that

$$(\sigma_{ab}(A) \cup \sigma_{ab}(B)) \setminus \sigma_{ab}(M_C) \subseteq S(A^*) \cap F(B),$$

where $F(B)$ is the Drazin spectrum of B (that is, $\lambda \notin F(B)$ if and only if $asc(B-\lambda) = dsc(B-\lambda) < \infty$), and $\sigma_{ab}(B) = \{\lambda \in \mathbb{C} : B - \lambda I$ is not an upper semi-Fredholm operator with finite ascent$\}$, which implies that

$$(\sigma_{ab}(A) \cup \sigma_{ab}(B)) \cup (S(A^*) \cap F(B)) = \sigma_{ab}(M_C) \cup (S(A^*) \cap F(B)). \tag{2}$$

The following example shows that neither the above claim nor equality (2) is always true.

Example 3.2. Let $H = K = \ell^2$ and define the operators A, B and C by

$$\begin{aligned} &A : H \to H, \qquad \{x_1, x_2, \ldots\} \mapsto \{x_1, 0, x_2, 0, \ldots\}, \\ &B : K \to K, \qquad \{x_1, x_2, \ldots\} \mapsto \{0, 0, 0, \ldots\}, \\ &C : K \to H, \qquad \{x_1, x_2, \ldots\} \mapsto \{0, x_1, 0, x_2, \ldots\}. \end{aligned}$$

Then

$$\sigma_{ab}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \qquad \sigma_{ab}(B) = \{0\}, \qquad \sigma_{ab}(M_C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \qquad S(A^*) \cap F(B) = \emptyset,$$
$$\sigma_{ab}(M_C) \ne \sigma_{ab}(A) \cup \sigma_{ab}(B).$$

So

$$(\sigma_{ab}(A) \cup \sigma_{ab}(B)) \setminus \sigma_{ab}(M_C) \not\subseteq S(A^*) \cap F(B).$$

Let A^* and $\sigma_p(A)$ denote the adjoint operator and the point spectrum of A, respectively, and $\sigma_g(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not Kato non-singular$\}$. In [1], the authors claimed that

$$\eta(\sigma_g(A) \cup \sigma_g(B)) = \eta(\sigma_g(M_C)).$$

More precisely,

$$\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B)) = \sigma_g(M_C) \cup W, \tag{3}$$

where W is the union of some holes in $\sigma_g(M_C)$ which happen to be the subsets of $\overline{\sigma_p(A^*)} \cup \sigma_p(B)$.

The following example shows that equality (3) is not always true.

Example 3.3. Let H be the direct sum of countably many copies of $\ell^2 := \ell^2(N)$. Thus, the elements of H are the sequences $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \ell^2$ and $\sum_{j=1}^{\infty} \|x_j\|^2 < \infty$. Put $K = \ell^2$. Let V be the forward shift on ℓ^2,

$$V : \ell^2 \to \ell^2, \qquad \{z_1, z_2, \ldots\} \mapsto \{0, z_1, z_2, \ldots\},$$

define the operators A, B and C by

$$\begin{aligned} &A : H \to H, && \{x_1, x_2, \ldots\} \mapsto \{Vx_1, Vx_2, \ldots\}, \\ &B : K \to K, && \{y_1, y_2, \ldots\} \mapsto \left\{0, y_1, \frac{1}{2}y_2, \frac{1}{3}y_3, \ldots\right\}, \\ &C : K \to H, && \{y_1, y_2, \ldots\} \mapsto \{y_1 e_1, y_2 e_1, \ldots\} \end{aligned}$$

where $e_1 = \{1, 0, 0, \ldots\}$. Consider the operator

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : H \oplus K \to H \oplus K.$$

Then we have

(i) $\sigma_g(M_C) = \sigma_g(A) = \{\lambda : |\lambda| = 1\}$,
(ii) $\sigma(B) = \sigma_g(B) = \{0\}$, $\sigma_p(B) = \emptyset$,
(iii) $\overline{\sigma_p(A^*)} \cap \sigma_p(B) = \emptyset$.

Therefore $W = (\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B))) \setminus \sigma_g(M_C) = \{0\}$. So W is just a point and thus not an open set from $\sigma_g(A) \cup \sigma_g(B) \cup (\overline{\sigma_p(A^*)} \cap \sigma_p(B))$ to $\sigma_g(M_C)$. This shows that equality (3) is not always true.

References

[1] M. Barraa, M. Boumazgour, On the perturbations of spectra of upper triangular operator matrices, J. Math. Anal. Appl. 347 (2008) 315–322.
[2] M. Barraa, M. Boumazgour, A note on the spectrum of an upper triangular operator matrix, Proc. Amer. Math. Soc. 131 (2003) 3083–3088.
[3] X.H. Cao, Browder spectra for upper triangular operator matrices, J. Math. Anal. Appl. 342 (2008) 477–484.
[4] X.H. Cao, M.Z. Guo, B. Meng, Semi-Fredholm spectrum and Weyl's theory for operator matrices, Acta Math. Sinica 22 (2006) 169–178.
[5] X.H. Cao, B. Meng, Essential approximate point spectra and Weyl's theorem for upper triangular operator matrices, J. Math. Anal. Appl. 304 (2005) 759–771.
[6] X.L. Chen, S.F. Zhang, H.J. Zhong, On the filling in holes problem of operator matrices, Linear Algebra Appl. 430 (2009) 558–563.
[7] D.S. Djordjević, Perturbations of spectra of operator matrices, J. Operator Theory 48 (2002) 467–486.
[8] S.V. Djordjević, Y.M. Han, Browder's theorem and spectral continuity, Glasg. Math. J. 42 (2000) 479–486.
[9] S.V. Djordjević, H. Zguitti, Essential point spectra of operator matrices though local spectral theory, J. Math. Anal. Appl. 338 (2008) 285–291.
[10] H.K. Du, J. Pan, Perturbation of spectrums of 2 × 2 operator matrices, Proc. Amer. Math. Soc. 121 (1994) 761–766.
[11] J.K. Han, H.Y. Lee, W.Y. Lee, Invertible completions of 2 × 2 upper triangular operator matrices, Proc. Amer. Math. Soc. 128 (1999) 119–123.
[12] I.S. Hwang, W.Y. Lee, The boundedness below of 2 × 2 upper triangular operator matrices, Integral Equations Operator Theory 39 (2001) 267–276.
[13] W.Y. Lee, Weyl spectra of operator matrices, Proc. Amer. Math. Soc. 129 (2000) 131–138.
[14] Y. Li, H.K. Du, The intersection of left and right essential spectra of 2 × 2 operator matrices, Bull. Lond. Math. Soc. 36 (2004) 811–819.
[15] Y. Li, H.K. Du, The intersection of essential approximate point spectra of operator matrices, J. Math. Anal. Appl. 323 (2006) 1171–1183.
[16] Y. Li, X.H. Sun, H.K. Du, The intersection of left(right) spectra of 2 × 2 upper triangular operator matrices, Linear Algebra Appl. 418 (2006) 112–121.
[17] Y. Li, X.H. Sun, H.K. Du, A note on the left essential approximate point spectra of operator matrices, Acta Math. Sinica 23 (2007) 2235–2240.
[18] E.H. Zerouali, H. Zguitti, Perturbation of spectra of operator matrices and local spectral theory, J. Math. Anal. Appl. 324 (2006) 992–1005.
[19] S.F. Zhang, H.J. Zhong, A note of Browder spectrum of operator matrices, J. Math. Anal. Appl. 344 (2008) 927–931.
[20] S.F. Zhang, H.J. Zhong, Q.F. Jiang, Drazin spectrum of operator matrices on the Banach space, Linear Algebra Appl. 429 (2008) 2067–2075.
[21] S.F. Zhang, Z.Y. Wu, H.J. Zhong, Continuous spectrum, point spectrum and residual spectrum of operator matrices, Linear Algebra Appl. 433 (2010) 653–661.
[22] S.F. Zhang, H.J. Zhong, J.D. Wu, Spectra of upper-triangular operator matrices, Acta Math. Sinica (Chin. Ser.) 54 (2011) 41–60 (in Chinese).
[23] S.F. Zhang, J.D. Wu, Samuel multiplicities and Browder spectrum of operator matrices, Oper. Matrices 6 (2012) 169–179.
[24] Y.N. Zhang, H.J. Zhong, L.Q. Lin, Browder spectra and essential spectra of operator matrices, Acta Math. Sinica (Chin. Ser.) 24 (2008) 947–954.